ZEROS OF RANDOM POLYNOMIALS ON $\mathbb{C}^m$

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ABSTRACT. For a regular compact set K in $\mathbb{C}^m$ and a measure μ on K satisfying the Bernstein-Markov inequality, we consider the ensemble $\mathcal{P}_N$ of polynomials of degree N, endowed with the Gaussian probability measure induced by $L^2(\mu)$. We show that for large N, the simultaneous zeros of m polynomials in $\mathcal{P}_N$ tend to concentrate around the Silov boundary of K; more precisely, their expected distribution is asymptotic to $N^m\mu_{eq}$, where μ_{eq} is the equilibrium measure of K. For the case where K is the unit ball, we give scaling asymptotics for the expected distribution of zeros as $N \to \infty$.

1. Introduction

A classical result due to Hammersley [Ha] (see also [SV]), loosely stated, is that the zeros of a random complex polynomial

$$f(z) = \sum_{j=0}^{N} c_j z^j \tag{1}$$

mostly tend towards the unit circle $|z| = 1$ as the degree $N \to \infty$, when the coefficients c_j are independent complex Gaussian random variables of mean zero and variance one. In this paper, we will prove a multivariable result (Theorem 3.1), a special case (Example 3.5) of which shows, loosely stated, that the common zeros of m random complex polynomials in $\mathbb{C}^m$,

$$f_k(z) = \sum_{|J|\le N} c_J^k \, z_1^{j_1} \cdots z_m^{j_m} \qquad \text{for } k = 1, \dots, m \ , \tag{2}$$

tend to concentrate on the product of the unit circles $|z_j| = 1$ $(j = 1, \dots, m)$ as $N \to \infty$, when the coefficients c_J^k are i.i.d. complex Gaussian random variables.

The following is our basic setting: We let K be a compact set in $\mathbb{C}^m$ and let μ be a Borel probability measure on K. We assume that K is non-pluripolar and we let V_K be its pluricomplex Green function. We also assume that K is regular (i.e., $V_K = V_K^*$) and that μ satisfies the Bernstein-Markov inequality (see §2). We give the space $\mathcal{P}_N$ of holomorphic polynomials of degree $\le N$ on $\mathbb{C}^m$ the Gaussian probability measure γ_N induced by the Hermitian inner product

$$(f, g) = \int_K f\bar{g}\, d\mu \ . \tag{3}$$

The Gaussian measure γ_N can be described as follows: We write $f = \sum_{j=1}^{d(N)} c_j p_j$, where $\{p_j\}$ is an orthonormal basis of $\mathcal{P}_N$ with respect to (3) and $d(N) = \dim \mathcal{P}_N = \binom{N+m}{m}$. Identifying

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$f \in \mathcal{P}_N$ with $c = (c_1, \dots, c_{d(N)}) \in \mathbb{C}^{d(N)}$, we have

$$d\gamma_N(s) = \frac{1}{\pi^{d(N)}} e^{-|c|^2} dc\,.$$

(The measure γ_N is independent of the choice of orthonormal basis $\{p_j\}$.) In other words, a random polynomial in the ensemble $(\mathcal{P}_N, \gamma_N)$ is a polynomial $f = \sum_j c_j\, p_j$, where the c_j are independent complex Gaussian random variables with mean 0 and variance 1.

Our main result, Theorem 3.1, gives asymptotics for the expected zero current of k i.i.d. random polynomials ($1 \le k \le m$). In particular, the expected distribution $\mathbf{E}(Z_{f_1,\dots,f_m})$ of simultaneous zeros of m independent random polynomials in $(\mathcal{P}_N, \gamma_N)$ has the asymptotics

$$\frac{1}{N^m}\, \mathbf{E}(Z_{f_1,\dots,f_m}) \to \mu_{eq} \quad weak^*\ , \tag{4}$$

where $\mu_{eq} = (\frac{i}{\pi}\partial\bar\partial V_K)^m$ is the equilibrium measure of K. Here, $\mathbf{E}(X)$ denotes the expected value of a random variable X.

The reader may notice from (4) that the distributions of zeros for the measures on $\mathcal{P}_N$ considered here are quite different from those of the $\mathrm{SU}(m+1)$ ensembles studied, for example, in [SZ1, SZ4, BSZ1, BSZ2, DS]. The Gaussian measure on the $\mathrm{SU}(m+1)$ polynomials is based on the inner product

$$\langle f, g\rangle_N = \int_{S^{2m+1}} F_N\, \overline{G_N},$$

where $F_N, G_N \in \mathbb{C}[z_0, z_1, \dots, z_m]$ denote the degree N homogenizations of f and g respectively. It follows easily from the $\mathrm{SU}(m+1)$-invariance of the inner product that the expected distribution of simultaneous zeros equals $\frac{N^m}{\pi^m}\omega^m$ (exactly), where ω is the Fubini-Study Kähler form (on $\mathbb{C}^m \subset \mathbb{CP}^m$). We note that, unlike (3), this inner product depends on N; indeed, $\|z^J\|_N^2 = \frac{m!(N-|J|)!j_1!\cdots j_m!}{(N+m)!}$ [SZ1, (30)].

In this paper, we also give scaling limits for the expected zero density in the case of the unit ball in $\mathbb{C}^m$ (Theorem 4.1). The problem of finding scaling limits for more general sets in $\mathbb{C}^m$ remains open. Another open problem is to establish the multivariable version of the following one variable result: For a regular subset $K \subset \mathbb{C}$, it is known (see [SZ1, Bl2]) that with probability one, a sequence $\{f_N\}_{N=1,2,\dots}$ of random polynomials of increasing degree satisfies:

$$\lim_{N\to\infty} \frac{1}{N} Z_{f_N} = \mu_{eq} \quad weak^*\ .$$

2. Background

We let $\mathcal{L}$ denote the Lelong class of plurisubharmonic (PSH) functions on $\mathbb{C}^m$ of at most logarithmic growth at ∞. That is

$$\mathcal{L} := \{u \in \mathrm{PSH}(\mathbb{C}^m) \mid u(z) \le \log^+\|z\| + O(1)\} \tag{5}$$

For K a compact subset of $\mathbb{C}^m$, we define its pluricomplex Green function $V_K(z)$ via

$$V_K(z) = \sup\{u(z) \mid u \in \mathcal{L},\ u \le 0 \ \text{ on } \ K\}. \tag{6}$$

We will assume K is regular, that is by definition, V_K is continuous on $\mathbb{C}^m$ (and so $V_K = V_K^*$, its uppersemicontinuous regularization). The function V_K is a locally bounded PSH function

on $\mathbb{C}^m$ and, in fact

$$V_K - \log^+ \|z\| = O(1) \; . \tag{7}$$

By a basic result of Bedford and Taylor [BT1] (see [Kl]), the complex Monge-Ampère operator $(dd^c)^m = (2i\partial\bar\partial)^m$ is defined on any locally bounded PSH function $\mathbb{C}^m$ and in particular on V_K. The equilibrium measure of K is defined by (see [Kl, Cor. 5.5.3])

$$\mu_{eq}(K) := \left(\frac{i}{\pi}\partial\bar\partial V_K\right)^m \tag{8}$$

Since V_K satisfies (7), it is a positive Borel measure, here normalized to have mass 1. The support of the measure $\mu_{eq}(K)$ is the Silov boundary of K for the algebra of entire analytic functions [BT2]. In one variable, i.e. $K \subset \mathbb{C}$, V_K is the Green function of the unbounded component of $\mathbb{C} \setminus K$ with a logarithmic pole at ∞, and $\mu_{eq}(K) = \frac{1}{2\pi}\Delta V_K$, where Δ is the Laplacian [Ra].

Let μ be a finite positive Borel measure on K. The measure μ is said to satisfy a Bernstein-Markov (BM) inequality, if, for each $\varepsilon > 0$ there is a constant $C = C(\varepsilon) > 0$ such that

$$\|p\|_K \le C e^{\varepsilon \deg(p)} \|p\|_{L^2(\mu)} \tag{9}$$

for all holomorphic polynomials p. Essentially, the BM inequality says that the L^2 norms and the sup norms of a sequence of holomorphic polynomials of increasing degrees are "asymptotically equivalent".

The question arises as to which measures actually satisfy the BM inequality. It is a result of Nguyen-Zeriahi [NZ] combined with [Kl, Cor. 5.6.7] that for K regular, $\mu_{eq}(K)$ satisfies BM. This fact is used in Examples 3.5–3.6. In [Bl1, Theorem 2.2], a "mass-density" condition for a measure to satisfy BM was given. (See also [BL].)

Our proof uses the *probabilistic Poincaré-Lelong formula* for the zeros of random functions (Proposition 2.1 below). Considering a slightly more general situation, we let $g_1, \dots, g_d$ be holomorphic functions with no common zeros on a domain $U \subset \mathbb{C}^m$. (We are interested in the case where $U = \mathbb{C}^m$ and $\{g_j\}$ is an orthonormal basis of $\mathcal{P}_N$ with respect to the inner product (3), as discussed above.) We let $\mathcal{F}$ denote the ensemble of random holomorphic functions of the form $f = \sum c_j\, g_j$, where the c_j are independent complex Gaussian random variables with mean 0 and variance 1. We consider the *Szegö kernel*

$$S_{\mathcal{F}}(z, w) = \sum_{j=1}^{d} g_j(z)\, \overline{g_j(w)} \; .$$

For the case where the g_j are orthonormal functions with respect to an inner product on $\mathcal{O}(U)$, $S_{\mathcal{F}}(z, w)$ is the kernel for the orthogonal projection onto the span of the g_j.

Under the assumption that the g_j have no common zeros, it is easily shown using Sard's theorem (or a variation of Bertini's theorem) that for almost all $f_1, \dots, f_k \in \mathcal{F}$, the differentials $df_1, \dots, df_k$ are linearly independent at all points of the zero set

$$\mathrm{loc}(f_1, \dots, f_k) := \{z \in U : f_1(z) = \cdots = f_k(z) = 0\} \; .$$

This condition implies that the complex hypersurfaces $\mathrm{loc}(f_j)$ are smooth and intersect transversely, and hence $\mathrm{loc}(f_1, \dots, f_k)$ is a codimension k complex submanifold of U. We

then let $Z_{f_1,\dots,f_k} \in \mathcal{D}'^{k,k}(U)$ denote the current of integration over $\mathrm{loc}(f_1,\dots,f_k)$:

$$(Z_{f_1,\dots,f_k},\varphi) = \int_{\mathrm{loc}(f_1,\dots,f_k)} \varphi\ , \qquad \varphi \in \mathcal{D}^{m-k,m-k}(U)\ .$$

We shall use the following Poincaré-Lelong formula from [SZ3, SZ4]:

PROPOSITION **2.1.** *The expected zero current of k independent random functions $f_1,\dots,f_k \in \mathcal{F}$ is given by*

$$\mathbf{E}(Z_{f_1,\dots,f_k}) = \left(\frac{i}{2\pi}\partial\bar\partial \log S_{\mathcal{F}}(z,z)\right)^k .$$

The proof follows by a verbatim repetition of the proof of Proposition 5.1 in [SZ3] (which gives the case where the g_j are normalized monomials with exponents in a Newton polytope). The codimension $k=1$ case was given in [SZ1] (for sections of holomorphic line bundles), and in dimension 1 by Edelman-Kostlan [EK]. (The formula also holds for infinite-dimensional ensembles; see [So, SZ4].) The general case follows from the codimension 1 case together with the fact that

$$\mathbf{E}(Z_{f_1,\dots,f_k}) = \mathbf{E}(Z_{f_1}) \wedge \cdots \wedge \mathbf{E}(Z_{f_k}) = \mathbf{E}(Z_f)^k\ , \tag{10}$$

which is a consequence of the independence of the f_j. The wedge product of currents is not always defined, but $Z_{f_1} \wedge \cdots \wedge Z_{f_k}$ is almost always defined (and equals $Z_{f_1,\dots,f_k}$ whenever the hypersurfaces $\mathrm{loc}(f_j)$ are smooth and intersect transversely), and a short argument given in [SZ3] yields (10). (In fact, the left equality of (10) holds for independent non-identically-distributed f_j, as proven in [SZ3].) We note that the expectations in (10) are smooth forms.

3. RANDOM POLYNOMIALS ON POLYNOMIALLY CONVEX SETS

THEOREM **3.1.** *Let μ be a Borel probability measure on a regular compact set $K \subset \mathbb{C}^m$, and suppose that (K,μ) satisfies the Bernstein-Markov inequality. Let $1 \le k \le m$, and let $(\mathcal{P}_N^k, \gamma_N^k)$ denote the ensemble of k-tuples of i.i.d. Gaussian random polynomials of degree $\le N$ with the Gaussian measure $d\gamma_N$ induced by $L^2(\mu)$. Then*

$$\frac{1}{N^k}\mathbf{E}_{\gamma_N^k}(Z_{f_1,\dots,f_k}) \to \left(\frac{i}{\pi}\partial\bar\partial V_K\right)^k \qquad weak^*, \quad as\ N \to \infty\ ,$$

where V_K is the pluricomplex Green function of K with pole at infinity.

To prove Theorem 3.1, we consider the Szegö kernels

$$S_N(z,w) := S_{(\mathcal{P}_N,\gamma_N)}(z,w) = \sum_{j=1}^{d(N)} p_j(z)\overline{p_j(w)}\ ,$$

where $\{p_j\}$ is an $L^2(\mu)$-orthonormal basis for $\mathcal{P}_N$. Our proof is based on approximating the extremal function V_K by the (normalized) logarithms of the Szegö kernels $S_N(z,z)$ (Lemma 3.4).

We begin by considering the polynomial suprema

$$\Phi_N^K(z) = \sup\{|f(z)| : f \in \mathcal{P}_N,\ \|f\|_K \le 1\}\ . \tag{11}$$

Since $\frac{1}{N}\log f \in \mathcal{L}$, for $f \in \mathcal{P}_N$, it is clear that $\frac{1}{N}\log \Phi_N^K \le V_K$, for all N. Pioneering work of Zaharjuta [Za] and Siciak [Si1, Si2] established the convergence of $\frac{1}{N}\log\Phi_N^K$ to V_K. The

uniform convergence when K is regular seems not to have been explicitly stated and we give the proof below.

LEMMA **3.2.** *Let K be a regular compact set in $\mathbb{C}^m$. Then*

$$\frac{1}{N}\log \Phi_N^K(z) \to V_K(z)$$

uniformly on compact subsets of $\mathbb{C}^m$.

Proof. We first note that $1 \leq \Phi_j \leq \Phi_j\Phi_k \leq \Phi_{j+k}$, for $j,k \geq 0$. By a result of Siciak [Si1] and Zaharjuta [Za] (see [Kl, Theorem 5.1.7]),

$$V_K(z) = \lim_{N\to\infty}\frac{1}{N}\log \Phi_N^K(z) = \sup_N \frac{1}{N}\log \Phi_N^K(z)\ , \tag{12}$$

for all $z \in \mathbb{C}^m$.

We use the regularity of K to show that the convergence is uniform: let

$$\psi_N = \frac{1}{N}\log \Phi_N^K \geq 0\ .$$

Thus for $N, k \geq 1$, $j \geq 0$, we have

$$Nk\,\psi_{Nk} + j\,\psi_j \leq (Nk+j)\,\psi_{Nk+j}\ .$$

Since $\psi_N \leq \psi_{Nk}$, we then obtain the inequality

$$\psi_{Nk+j} \geq \frac{Nk}{Nk+j}\psi_N + \frac{j}{Nk+j}\psi_j \geq \frac{Nk}{Nk+j}\psi_N\ . \tag{13}$$

Fix $\varepsilon > 0$. For each $a \in \mathbb{C}^m$, we choose $N_a \in \mathbb{Z}^+$ such that

$$V_K(a) - \psi_{N_a}(a) < \varepsilon \qquad \text{and} \qquad \frac{V_K(a)}{N_a} < \varepsilon\ ,$$

and then choose a neighborhood U_a of a such that

$$|V_K(z) - V_K(a)| < \varepsilon, \quad \psi_{N_a}(z) \geq \psi_{N_a}(a) - \varepsilon, \quad \frac{V_K(z)}{N_a} < \varepsilon, \qquad \text{for} \ \ z \in U_a\ .$$

Now let $N \geq N_a^2$, and write $N = N_a k + j$, where $k \geq N_a,\ 0 \leq j < N_a$. By (12)–(13), we have

$$0 \leq V_K - \psi_N \leq V_K - \frac{N_a k}{N_a k + j}\psi_{N_a} \leq V_K - \frac{N_a}{N_a+1}\psi_{N_a} \leq V_K - \psi_{N_a} + \frac{1}{N_a+1}V_K\ . \tag{14}$$

Hence, for all $N \geq N_a^2$ and for all $z \in U_a$, we have

$$\begin{aligned} 0 \ \leq\ V_K(z) - \psi_N(z) &< V_K(z) - \psi_{N_a}(z) + \varepsilon \\ &= [V_K(a) - \psi_{N_a}(a)] + [V_K(z) - V_K(a)] + [\psi_{N_a}(a) - \psi_{N_a}(z)] + \varepsilon \\ &< 4\varepsilon\ . \end{aligned} \tag{15}$$

Hence for each compact $A \subset \mathbb{C}^m$, we can cover A with finitely many U_{a_i}, so that we have by (15),

$$\|V_K - \psi_N\|_A \leq 4\varepsilon \qquad \forall\ N \geq \max_i N_{a_i}^2\ .$$

□

LEMMA **3.3.** *For all $\varepsilon > 0$, there exists $C = C_\varepsilon > 0$ such that*

$$\frac{1}{d(N)} \le \frac{S_N(z,z)}{\Phi_N^K(z)^2} \le C\, e^{\varepsilon N} d(N).$$

Proof. Let $f \in \mathcal{P}_N$ with $\|f\|_K \le 1$. Then

$$\begin{aligned} |f(z)| &= \left|\int_K S_N(z,w) f(w)\, d\mu(w)\right| \le \int_K |S_N(z,w)|\, d\mu(w) \\ &\le \int_K S_N(z,z)^{\frac12} S_N(w,w)^{\frac12}\, d\mu(w) = S_N(z,z)^{\frac12}\, \|S_N(w,w)^{\frac12}\|_{L^1(\mu)} \\ &\le S_N(z,z)^{\frac12}\, \|1\|_{L^2(\mu)}\, \|S_N(w,w)^{\frac12}\|_{L^2(\mu)} = S_N(z,z)^{\frac12}\, d(N)^{\frac12}\ . \end{aligned}$$

Taking the supremum over $f \in \mathcal{P}_N$ with $\|f\|_K \le 1$, we obtain the left inequality of the lemma.

To verify the right inequality, we let $\{p_j\}$ be a sequence of $L^2(\mu)$-orthonormal polynomials, obtained by applying Gram-Schmid to a sequence of monomials of non-decreasing degree, so that $\{p_1, \dots p_{d(N)}\}$ is an orthonormal basis of $\mathcal{P}_N$ (for each $N \in \mathbb{Z}^+$). By the Bernstein-Markov inequality (9), we have

$$\|p_j\|_K \le C\, e^{\varepsilon \deg p_j}$$

and hence

$$|p_j(z)| \le \|p_j\|_K\, \Phi^K_{\deg p_j}(z) \le C\, e^{\varepsilon \deg p_j}\, \Phi^K_{\deg p_j}(z) \le C\, e^{\varepsilon N}\, \Phi_N^K(z)\ , \quad \text{for } \ j \le d(N).$$

Therefore,

$$S_N(z,z) = \sum_{j=1}^{d(N)} |p_j(z)|^2 \le d(N)\, C^2\, e^{2\varepsilon N} \Phi_N^K(z)^2\ .$$

□

LEMMA **3.4.** *Under the hypotheses of Theorem 3.1, we have*

$$\frac{1}{2N} \log S_N(z,z) \to V_K(z)$$

uniformly on compact subsets of $\mathbb{C}^m$.

Proof. Let $\varepsilon > 0$ be arbitrary. Recalling that $d(N) = \binom{N+m}{m}$, we have by Lemma 3.3,

$$-\frac{m}{N}\log(N+m) \le \frac{1}{N}\log\left(\frac{S_N(z,z)}{\Phi_N^K(z)^2}\right) \le \frac{\log C}{N} + \varepsilon + \frac{m}{N}\log(N+m)\ .$$

Since $\varepsilon > 0$ is arbitrary, we then have

$$\frac{1}{N}\log\left(\frac{S_N(z,z)}{\Phi_N^K(z)^2}\right) \to 0\ . \tag{16}$$

The conclusion follows from Lemma 3.2 and (16). □

Proof of Theorem 3.1: It follows from Lemma 3.4 and the fact that the complex Monge-Àmpere operator is continuous under uniform limits [BT1],

$$\left(\frac{i}{2\pi N}\partial\bar\partial \log S_N(z,z)\right)^k \to \left(\frac{i}{\pi}\partial\bar\partial V_K(z)\right)^k \qquad weak^*\ .$$

The conclusion then follows from Proposition 2.1. $\square$

Example 3.5. Let K be the unit polydisk in $\mathbb{C}^m$. Then $V_K = \max_{j=1}^m \log^+ |z_j|$, the Silov boundary of K is the product of the circles $|z_j| = 1$ $(j = 1, \dots, m)$, and $d\mu_{eq} = (\frac{1}{2\pi})^m d\theta_1 \cdots d\theta_m$ where $d\theta_j$ is the angular measure on the circle $|z_j| = 1$.

The monomials $z^J := z_1^{j_1} \cdots z_m^{jm}$, for $|J| \leq N$, form an orthonormal basis for $\mathcal{P}_N$. A random polynomial in the ensemble is of the form

$$f(z) = \sum_{|J| \leq N} c_J z^J$$

where the c_J are independent complex Gaussian random variables of mean zero and variance one. By Theorem 3.1, $\mathbf{E}_{\gamma_N^k}(Z_{f_1,\dots,f_m}) \to (\frac{1}{2\pi})^m d\theta_1 \cdots d\theta_m$ weak*, as $N \to \infty$. In particular, the common zeros of m random polynomials tend to the product of the unit circles $|z_j| = 1$ for $j = 1, \dots, m$.

Example 3.6. Let K be the unit ball $\{\|z\| \leq 1\}$ in $\mathbb{C}^m$. Then the Silov boundary of K is its topological boundary $\{\|z\| = 1\}$, $V_K(z) = \log^+ \|z\|$, and μ_{eq} is the invariant hypersurface measure on $\|z\| = 1$ normalized to have total mass one.

4. Scaling limit zero density for orthogonal polynomials on S^{2m-1}

Examples 3.5 and 3.6 both reduce to the unit disk in the one variable case. In that case, detailed scaling limits are known (see, for example, [IZ]). For a more general compact set $K \subset \mathbb{C}$ with an analytic boundary, scaling limits are found in [SZ2].

In this section, we consider the case where $K = \{z \in \mathbb{C}^m : \|z\| \leq 1\}$ is the unit ball and μ is its equilibrium measure, i.e. invariant measure on the unit sphere S^{2m-1}. We have the following scaling asymptotics for the expected distribution of zeros of m random polynomials orthonormalized on the sphere:

Theorem **4.1.** *Let $(\mathcal{P}_N^m, \gamma_N^m)$ denote the ensemble of m-tuples of i.i.d. Gaussian random polynomials of degree $\leq N$ with the Gaussian measure $d\gamma_N$ induced by $L^2(S^{2m-1}, \mu)$, where μ is the invariant measure on the unit sphere $S^{2m-1} \subset \mathbb{C}^m$. Then*

$$\mathbf{E}_{\gamma_N^m}(Z_{f_1,\dots,f_m}) = D_N \left(\log \|z\|^2\right) \left(\frac{i}{2}\partial\bar{\partial}\|z\|^2\right)^m ,$$

where

$$\frac{1}{N^{m+1}} D_N\left(\frac{u}{N}\right) = \frac{1}{\pi^m} F_m''(u)\, F_m'(u)^{m-1} + O\left(\frac{1}{N}\right) , \quad F_m(u) = \log\left[\frac{d^{m-1}}{du^{m-1}}\left(\frac{e^u - 1}{u}\right)\right].$$

Proof. We write

$$z^J = z_1^{j_1} \cdots z_m^{j_m} , \qquad z = (z_1, \dots, z_m),\ J = (j_1, \dots, j_m) .$$

An easy computation yields

$$\int_{S^{2m-1}} |z^J|^2 \, d\mu(z) = \frac{(m-1)! j_1! \cdots j_m!}{(|J| + m - 1)!} = \frac{1}{\binom{|J|+m-1}{m-1}\binom{|J|}{J}} , \tag{17}$$

where

$$|J| = j_1 + \cdots + j_m , \qquad \binom{|J|}{J} = \frac{|J|!}{j_1! \cdots j_m!} .$$

Thus an orthonormal basis for $\mathcal{P}_N$ on S^{2m-1} is:

$$\varphi_J(z) = \binom{|J|+m-1}{m-1}^{\frac{1}{2}} \binom{|J|}{J}^{\frac{1}{2}} z^J \,, \qquad |J| \le N \,. \tag{18}$$

We have

$$\begin{aligned} S_N(z,z) = \sum_{|J|\le N} |\varphi_J(z)|^2 &= \sum_{k=0}^{N} \binom{k+m-1}{m-1} \sum_{|J|=k} \binom{k}{J} |z_1|^{2j_1} \cdots |z_m|^{2j_m} \\ &= \sum_{k=0}^{N} \binom{k+m-1}{m-1} \|z\|^{2k} \,. \end{aligned}$$

Hence

$$S_N(z,z) = g_N(\|z\|^2) \,, \quad \text{where } \ g_N(x) = \sum_{k=0}^{N} \binom{k+m-1}{m-1} x^k \,. \tag{19}$$

We note that

$$g_N = \frac{1}{(m-1)!} G_N^{(m-1)} \,, \quad \text{where } \ G_N(x) = \frac{1-x^{N+m}}{1-x} \,.$$

We denote by $O(\frac{1}{N})$ any function $\lambda(N,u) = \lambda_N(u) : \mathbb{Z}^+ \times \mathbb{R} \to \mathbb{R}$ satisfying:

$$\forall R>0, \ \forall j \in \mathbb{N}, \ \exists C_{Rj} \in \mathbb{R}^+ \quad \text{such that } \sup_{|u|<R} |\lambda_N^{(j)}(u)| < \frac{C_{Rj}}{N} \,. \tag{20}$$

We note that

$$N \log\left(1 + \frac{u}{N}\right) = u + u^2\, O\left(\frac{1}{N}\right) \qquad (\text{for } \ |u| < N) \,,$$

and hence

$$\left(1 + \frac{u}{N}\right)^N = e^u + u^2\, O\left(\frac{1}{N}\right) \,.$$

Thus we have

$$\frac{1}{N} G_N\left(1 + \frac{u}{N}\right) = \frac{e^u - 1}{u} + O\left(\frac{1}{N}\right) \,. \tag{21}$$

Hence

$$\frac{1}{N^m} g_N\left(1 + \frac{u}{N}\right) = \frac{1}{(m-1)!} \frac{d^{m-1}}{du^{m-1}} \left(\frac{e^u - 1}{u}\right) + O\left(\frac{1}{N}\right) \,. \tag{22}$$

Therefore

$$\log\left[\frac{(m-1)!}{N^m} g_N\left(1 + \frac{u}{N}\right)\right] = F_m(u) + O\left(\frac{1}{N}\right) \,, \tag{23}$$

where F_m is given in the statement of the theorem.

Since the zero distribution is invariant under the $\mathrm{SO}(2m)$-action on $\mathbb{C}^m$, we can write

$$\mathbf{E}_{\gamma_N^m}(Z_{f_1,\dots,f_m}) = D_N\left(\log \|z\|^2\right) \left(\frac{i}{2}\partial\bar\partial \|z\|^2\right)^m \,. \tag{24}$$

Then $D_N(\frac{u}{N})$ is the density at the point

$$z^N := \left(\frac{1}{\sqrt{m}}\, e^{u/2N}, \dots, \frac{1}{\sqrt{m}}\, e^{u/2N}\right) \in \mathbb{C}^m\,, \qquad \|z^N\|^2 = e^{u/N}\,.$$

We shall compute using the local coordinates $\zeta_j = \rho_j + i\theta_j = \log z_j$. Let

$$\Omega = \left(\frac{i}{2}\partial\bar\partial \sum |\zeta_j|^2\right)^m \,.$$

By Proposition 2.1 and (19), we have

$$\mathbf{E}_{\gamma_N^m}(Z_{f_1,\dots,f_m}) = \left(\frac{1}{2\pi}\right)^m \det\left(\frac{1}{2}\,\frac{\partial^2}{\partial\rho_j\partial\rho_k}\log g_N\left(\sum e^{2\rho_j}\right)\right)\Omega\,. \tag{25}$$

We note that

$$\Omega = m^m\left[1 + O\left(\frac{1}{N}\right)\right]\left(\frac{i}{2}\partial\bar\partial\|z\|^2\right)^m \qquad \text{at the point } \ z^N\,. \tag{26}$$

We let $\mathbf{1}$ denote the $m \times m$ matrix all of whose entries are equal to 1 (and we let I denote the $m \times m$ identity matrix). By (23) and (25)–(26), we have

$$\begin{aligned}
D_N\left(\frac{u}{N}\right) &= \left(\frac{m}{2\pi}\right)^m\left[1 + O\left(\frac{1}{N}\right)\right] \\
&\quad \times \det\left(2\,m^{-2}\,e^{2u/N}(\log g_N)''(e^{u/N})\,\mathbf{1} + 2\,m^{-1}\,e^{u/N}(\log g_N)'(e^{u/N})\,I\right) \\
&= \frac{1}{\pi^m}\left[1 + O\left(\frac{1}{N}\right)\right]\det\left(m^{-1}N^2\,F_m''(u)\,\mathbf{1} + N\,F_m'(u)\,I\right)\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{N^{m+1}}\,D_N\left(\frac{u}{N}\right) &= \frac{1}{N^{m+1}\,\pi^m}\left[1 + O\left(\frac{1}{N}\right)\right] \\
&\quad \times\left\{\left[N\,F_m'(u)\right]^m + \ m\left[m^{-1}N^2\,F_m''(u)\right]\left[N\,F_m'(u)\right]^{m-1}\right\} \\
&= \frac{1}{\pi^m}\,F_m''(u)\,F_m'(u)^{m-1} + O\left(\frac{1}{N}\right)\,.
\end{aligned}$$

□

Remark: There is a similarity between the scaling asymptotics of Theorem 4.1 and that of the one-dimensional SU(1,1) ensembles in [BR] with the norms $\|z^j\| = \binom{L-1+j}{j}^{-1/2}$, for $L \in \mathbb{Z}^+$. Then the expected distribution of zeros of random SU(1,1) polynomials of degree N has the asymptotics [BR, Th. 2.1]:

$$\mathbf{E}_N(Z_f) = \widetilde{D}_N\left(\log|z|^2\right)\,\frac{i}{2}dz\wedge d\bar z\,,$$

where (in our notation)

$$\frac{1}{N^2}\widetilde{D}_N\left(\frac{u}{N}\right) = \frac{1}{\pi}\,F_{L-1}''(u) + O\left(\frac{1}{N}\right)\,.$$

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